An Analytical Model for Entropy Noise of Subsonic Nozzle Flow

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An analytical model was developed for the evaluation of entropy noise generated in a low Mach number nozzle flow. The acoustic intensity radiated from the nozzle exit was obtained in closed form. Correlations among upstream temperature, pressure, and velocity fluctuations are required for calculation of radiated noise. The mean flow and the flow inhomogeneities were assumed to be quasi-one-dimensional, and an exponential nozzle was selected to simplify the analysis. Results show that the upstream fluctuation of temperature is an important source of nozzle entropy noise and the noise intensity is roughly proportional to the nozzle contraction rate.

Introduction

THE noise generated by a model nozzle with a low Mach number hot jet frequently includes upstream noise. The upstream noise sources include the burner, valves, and the duct flow. In the case of an engine operated at low power settings, the core noise level usually exceeds that of the pure jet noise. One of the core noise sources can be called "nozzle entropy noise." The nozzle entropy is generated from upstream inhomogeneities (e.g., hot and cold temperature spots) in the nozzle flow. The difference in acceleration of these inhomogeneities inside the nozzle causes the pressure and velocity to fluctuate at the nozzle exit, and therefore generates noise.

Efforts in recent years have been made to study this noise source. Marble[1] considered a one-dimensional "disk" model in which the inhomogeneities were much larger than nozzle length. He obtained the downstream pressure disturbances in terms of given upstream disturbances. Both the subsonic and supersonic cases were analyzed. Candel[2] analyzed the entropy noise of a sonic nozzle with fixed characteristics (Mach) lines at the nozzle exit while the flow was perturbed. A subsonic case was solved numerically by Bohn.[3] All the preceding works used a one-dimensional wave assumption as the downstream matching boundary condition. This boundary condition does not allow the noise to radiate at all angles from the nozzle exit, except along the jet axis, and therefore, is not used in the present analysis. A general analysis of the entropy noise radiated from a nozzle of a low Mach number jet was made by Ffowcs Williams and Howe.[4] They considered compact source at the nozzle exit. Inhomogeneities convected through a nozzle were also analyzed. Application of these models to predict engine and model entropy noise requires the knowledge of detailed information of upstream disturbances, which is difficult to obtain.

The model developed in this report allows the calculation of entropy noise to be made using upstream correlations which can be more easily obtained than the individual disturbances. This analysis emphasizes low-velocity hot jets rather than high velocity jets, since entropy noise is of little practical interest at high jet velocities, where the jet noise is dominating. It is expected that the size of upstream hot or cold "spots" are not necessarily much larger than the nozzle length. Therefore, a finite nozzle length was used in the analysis.

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A quasi-one-dimensional flow model is chosen for the present analysis. An exponential convergent nozzle is used to simplify the algebra so that an explicit closed-form solution can be obtained. The results should be applicable to most smooth convergent nozzles.

Analysis

The following quasi-one-dimensional equations for an unsteady, inviscid, nonheat-conducting ideal gas are used.

$$\frac{\partial \rho}{\partial t}+\frac{1}{A}\frac{\partial(\rho U A)}{\partial x}=0 \tag{1}$$

$$\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+\frac{1}{\rho}\frac{\partial P}{\partial x}=0 \tag{2}$$

$$\frac{\partial s}{\partial t}+U\frac{\partial S}{\partial x}=0 \tag{3}$$

$$P-R\rho T=0 \tag{4}$$

where the specific entropy s satisfies

$$\mathrm{d}s=R\left[\frac{1}{\gamma-1}\frac{\mathrm{d}T}{T}+\frac{P}{T}\mathrm{d}\left(\frac{T}{P}\right)\right] \tag{5}$$

ρ, U, P, and T are the gas density, velocity, pressure, and temperature, respectively. A is the nozzle cross-sectional area; $\gamma=c_p/c_v$ is the ratio of specific heats; R is the ideal gas constant; and t and x are time and axial coordinates with $x=0$ at the nozzle exit and positive in the jet direction. It is assumed that the square of jet Mach number M_j is small. A small perturbation parameter $\epsilon=M_j^2$ is used to linearize the problem. The unsteady part of the flow is caused by disturbances and flow inhomogeneities which are small in comparison with the mean flow quantities. An exponential nozzle ($A=A_j e^{-a\xi}$) is used to simplify the formulation of this entropy noise model.

With the assumptions of low Mach number flow and small disturbance, the steady flow solution is obtained. A small perturbation problem is formulated for the unsteady disturbances by assuming the following dimensionless forms:

$$\frac{\rho}{\rho_j}=1+\frac{\epsilon}{2}(1-e^{2a\xi})+\rho_0(\xi,\tau)+\epsilon\rho_1(\xi,\tau) \tag{6a}$$

$$\frac{P}{P_j}=1+\epsilon\frac{\gamma}{2}(1-e^{2a\xi})+p_0(\xi,\tau)+\epsilon p_1(\xi,\tau) \tag{6b}$$

$$\frac{T}{T_j}=1+\epsilon\frac{\gamma-1}{2}(1-e^{2a\xi})+T_0(\xi,\tau)+\epsilon T_1(\xi,\tau) \tag{6c}$$

$$\frac{U}{U_j}=e^{a\xi}\left[1-\frac{\epsilon}{2}(1-e^{2a\xi})\right]+u_0(\xi,\tau)+\epsilon u_1(\xi,\tau) \tag{6d}$$

where j denotes the jet exit condition that is used as reference, $\xi=x/\ell$, and $\tau=tU_j/\ell$, with ℓ as the nozzle length. The preceding perturbation scheme makes use of the small Mach number exponential nozzle flow which is different from that of Candel's in that the perturbation was solved for Tsien's nozzle[5] in which the velocity increases linearly with axial distance. The zeroth-order solution ρ_0, p_0, T_0, and u_0 and the first-order solution ρ_1, p_1, T_1, and u_1 are given in Appendix A.

It is to be noticed that in this analysis all the boundary conditions are assumed to be given at an upstream location. Since this is a totally subsonic problem, the upstream flow can be influenced by the downstream conditions. Therefore, the upstream boundary conditions have to be obtained by measurements. It cannot be arbitrarily assigned.

The perturbation solutions require a set of upstream boundary conditions in perturbation forms that cannot be obtained practically. Fortunately, the calculation of entropy noise radiated from the nozzle exit does not require this set of detailed boundary conditions.

Acoustic Intensity at the Nozzle Exit

An investigation of zeroth-order solution shows that p_0 is the pressure caused by the unsteady operation of a nozzle test rig or an engine. The unsteady combustion can cause upstream pressure fluctuations. It is the direct combustion noise and it is not included in this analysis. It is assumed in this analysis that p_0 vanishes under steady operating conditions, which causes the u_0 to vanish also; i.e.,

$$p_0=u_0=0 \tag{7}$$

for steady operation of nozzle rigs or engines. The calculation of nozzle entropy noise under this condition is simplified considerably and a set of boundary conditions can be obtained. From Eq. (7), the problem is reduced to that of finding the entropy noise caused by upstream zeroth-order temperature disturbances.

The time average of the acoustic intensity radiated from the nozzle exit ($\xi=0$) is evaluated by the correlation

$$W(\tau')=\lim_{\tau_{\rm int}\to\infty}\frac{\tau^2P_jU_j}{\tau_{\rm int}}\int_{-\epsilon_{\rm int}/2}^{\tau_{\rm int}/2}p_1(0,\tau)u_1(0,\tau+\tau')\,\mathrm{d}\tau \tag{8}$$

at $\tau'=0$ where $\tau_{\rm int}$ is the dimensionless integration time.

The acoustic intensity spectrum is the real part of $\bar{W}(\Omega)$, where

$$\bar{W}/\Omega)=\frac{1}{2\pi}\left(\frac{\ell}{U_j}\right)\int_{-\infty}^{\infty}W(\tau')e^{i\Omega\tau'}\mathrm{d}\tau'$$
$$=\epsilon^2P_jU_j\left(\frac{\ell}{U_j}\right)\frac{2\pi}{\tau_{\rm int}}\bar{p}_1^*(0;\Omega)\bar{u}_1(0;\Omega) \tag{9}$$

is obtained by convolution of the Fourier transform. It is found from the first-order solution in Appendix A that

$$\bar{p}_1(0;\Omega)=B\bar{T}_0(\xi_0;\Omega)+\bar{p}_1(\xi_0;\Omega) \tag{10}$$

$$\bar{u}_1(0;\Omega)=E\bar{T}_0(\xi_0;\Omega)+\frac{i\Omega}{\gamma a}(e^{-a\xi_0}-1)\bar{p}_1(\xi_0;\Omega)$$
$$+e^{a\xi_0}\bar{u}_1(\xi_0;\Omega) \tag{11}$$

where B and E are given in Appendix B. From Eq. (9), the acoustic intensity spectrum as a function of frequency f can be written as

$$\begin{aligned}\bar{W}(f)_r=\epsilon^2P_jU_j\Big\{&(B_rE_r+B_iE_i)(\bar{T}_0^*\bar{T}_0)\\&+\left[\frac{\Omega}{\gamma a}(e^{-a\xi_0}-1)B_i+E_r\right](\bar{p}_1^*\bar{T}_0)_r\\&+\left[\frac{\Omega}{\gamma a}(e^{-a\xi_0}-1)B_r-E_i\right](\bar{p}_1^*\bar{T}_0)_i\\&+e^{-a\xi_0}(\bar{p}_1^*\bar{u}_1)_r+e^{-a\xi_0}B_r(\bar{T}_0^*-\bar{u}_1)_r+e^{-a\xi_0}B_i(\bar{T}_0^*\bar{u}_1)_i\Big\}\end{aligned} \tag{12}$$

where

$$(\bar{T}_0^*\bar{T}_0)=\frac{\ell}{U_j}\frac{2\pi}{\tau_{\rm int}}\bar{T}_0^*(\xi_0;\Omega)\bar{T}_0(\xi_0;\Omega)$$

$$(\bar{p}_1^*\bar{T}_0)=\frac{\ell}{U_j}\frac{2\pi}{\tau_{\rm int}}\bar{p}_1^*(\xi_0;\Omega)\bar{T}_0(\xi_0;\Omega)$$

$$(\bar{p}_1^*\bar{u}_1)=\frac{\ell}{U_j}\frac{2\pi}{\tau_{\rm int}}\bar{p}_1^*(\xi_0;\Omega)\bar{u}_1(\xi_0;\Omega)$$

$$(\bar{T}_0^*\bar{u}_1)=\frac{\ell}{U_j}\frac{2\pi}{\tau_{\rm int}}\bar{T}_0^*(\xi_0;\Omega)\bar{u}_1(\xi_0;\Omega)$$

are correlations of upstream dimensionless disturbances and the subscripts r and i denote the real and imaginary parts, respectively. These correlations are measured as functions of frequency f with any reasonable bandwidth. In this case, the dimensionless frequency $\Omega=2\pi f\ell/U_j$ is calculated for each center frequency in Eq. (12). The intensity level for each frequency band is given by

$$IL(f)=10\log_{10}[\bar{W}(f)_r/10^{-12}\text{watt}/m^2]\text{d}B \tag{13}$$

at the nozzle exit.

The entropy noise calculation requires the upstream correlations among upstream temperature, pressure, and velocity fluctuations. A brief description is now given.

$(\bar{T}_0^*\bar{T}_0)$ is the autocorrelation of upstream temperature fluctuations. Temperature fluctuations are considered to be the main mechanism of entropy noise generation in nozzle flow. One can use the autocorrelation of upstream total-temperature fluctuations instead of the autocorrelation of the static temperature fluctuations, since the upstream flow Mach number is small. This substitution simplifies the measurements. On the other hand, if the temperature fluctuation is not the dominating source of the upstream inhomogeneities, the nozzle entropy noise can be ignored among other noise sources. The correlations $(\bar{p}_1^*\bar{T}_0)$ and $(\bar{T}_0^*\bar{u}_1)$ are expected to be small because the convection speed of T_0 and the propagation speed of p_1 and u_1 are different.

$(\bar{p}_1^*\bar{u}_1)_r$ comes from the propagating part of upstream noise. The coefficient $e^{-a\xi_0}$ of $(\bar{p}_1^*\bar{u}_1)_r$ shows that the total power of the direct upstream noise is conserved through the nozzle flow, since the exponential nozzle cross-section area decreases in the flow direction by the factor $e^{-a\xi}$.

High- and Low-Frequency Limits

For the high-frequency limit, $\Omega\to\infty$, the coefficients for $(\bar{T}_0^*\bar{T}_0)$ in Eq. (12) take the following limits

$$B\sim i\gamma a\frac{1}{\Omega}\left\{e^{a\xi_0}-\exp\left[\frac{i\Omega}{a}(e^{-a\xi_0}-1)\right]\right\}\sim 0$$

$$E\sim-\frac{\gamma}{2}+e^{a\xi_0}-\left(1-\frac{\gamma}{2}\right)e^{2a\xi_0}$$

The high-frequency entropy noise generated by finite $(\bar{T}_0^*\bar{T}_0)$ and $(\bar{T}_0^*\bar{u}_I)$ vanishes at the rate $O(1/\Omega)$, although both correlations vanish at high frequencies in practical cases. It is expected that $(\bar{T}_0^*\bar{u}_I)$ has an even higher decreasing rate than that of $(\bar{T}_0^*\bar{T}_0)$ for increasing frequencies. The coefficients for $(\bar{p}_I^*\bar{T}_0)$ take the following limits:

$$\frac{\Omega}{\gamma a}(e^{-a\xi_0}-1)B_i+E_r$$

$$\sim\left(1-\frac{\gamma}{2}\right)(1-e^{2a\xi_0})-(e^{-a\xi_0}-1)\cos\left[\frac{\Omega}{a}(e^{-a\xi_0}-1)\right]$$

$$\frac{\Omega}{\gamma a}(e^{-a\xi_0}-1)B_r-E_i\sim(e^{-a\xi_0}-1)\sin\left[\frac{\Omega}{a}(e^{-a\xi_0}-1)\right]$$

These coefficients are bounded oscillating functions for $(\bar{p}_I^*\bar{T}_0)$. In practical cases, $(\bar{p}_I^*\bar{T}_0)$ vanishes rapidly for high frequencies. One can conclude that a nozzle does not generate high-frequency entropy noise (upstream noise is not included).

For the low-frequency limit, $\Omega\to 0$, the limits of the coefficients are

$$B\sim-\gamma a\xi_0$$

$$E\sim-(\gamma-1)(1-e^{2a\xi_0})+a\xi_0$$

In this case, $BE(\bar{T}_0^*\bar{T}_0)$ is negative. The low-frequency entropy noise, generated from $(\bar{T}_0^*\bar{T}_0)$ is propagating upstream. The temperature autocorrelation term compensates other terms in the calculation of radiated acoustic intensity and, therefore, attenuates the very low-frequency entropy noise. One can expect that a nozzle does not radiate very low-frequency entropy noise if the temperature autocorrelation is the main mechanism of entropy noise generation.

Results

The calculation of entropy noise requires the knowledge of upstream correlations $(\bar{T}_0^*\bar{T}_0)$, $(\bar{p}_I^*\bar{T}_0)$, $(\bar{T}_0^*\bar{u}_I)$, and $(\bar{p}_I^*\bar{u}_I)$. These correlations are not available at present, but their coefficients in Eq. (12) can be calculated. The coefficients of the upstream correlation can be considered as operators that transform the given input into noise radiated from the nozzle exit.

Calculation of these coefficients is made by assigning a value of 1 to all the dimensionless correlations for all the frequencies. Figure 1 is a plot of $B_rE_r+B_iE_i$, which is the coefficient of the upstream autocorrelation of temperature disturbances $(\bar{T}_0^*\bar{T}_0)$. It vanishes in the high-frequency limit, and becomes negative for low frequencies, as shown by the analysis. The oscillation of the curve means that the positive values generate downstream noise and the negative values generate upstream noise.

Figure 2 shows the coefficients for both the real part and the imaginary part of the pressure-temperature correlation term. The oscillations are bounded in the high-frequency limits. However, the correlation $(\bar{p}_I^*\bar{T}_0)$ may vanish before the limiting cases are needed. Contrary to the results for the autocorrelation $(\bar{T}_0^*\bar{T}_0)$, the direction of the noise propagation for $(\bar{p}_I^*\bar{T}_0)$ cannot be predicted by the sign of the coefficients, since the real or the imaginary part of $(\bar{p}_I^*\bar{T}_0)$ may not necessarily be always positive or negative. The direction of noise propagation is determined by both $(\bar{p}_I^*\bar{T}_0)$ and its coefficients.

The coefficients of the temperature-velocity correlation term are shown in Fig. 3. The oscillations vanish in the high-frequency limit and thus will not generate high-frequency noise.

The calculation of entropy noise generated by autocorrelation $(\bar{T}_0^*\bar{T}_0)$ is illustrated in the following example:

Example 1

Consider an exponential nozzle of length $\ell=0.4$ m, jet velocity $U_j=400$ m/s and Mach number $M_j=0.75$ (pressure ratio = 1.44, total temperature 800 K). From Fig. 1 (suppose

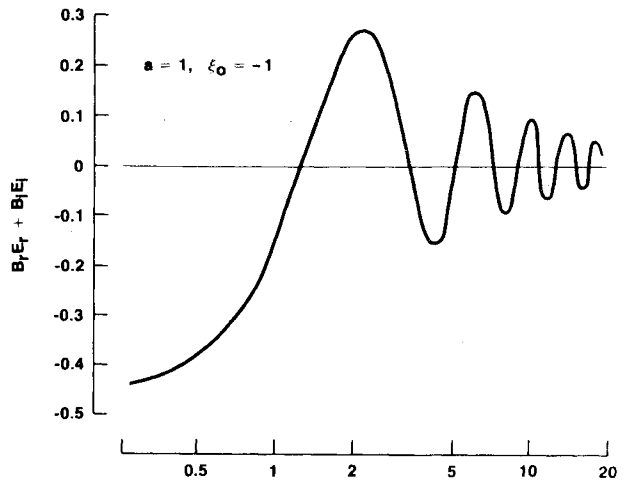

Fig. 1 Coefficient of $(\bar{T}_0^*\bar{T}_0)$ correlations.

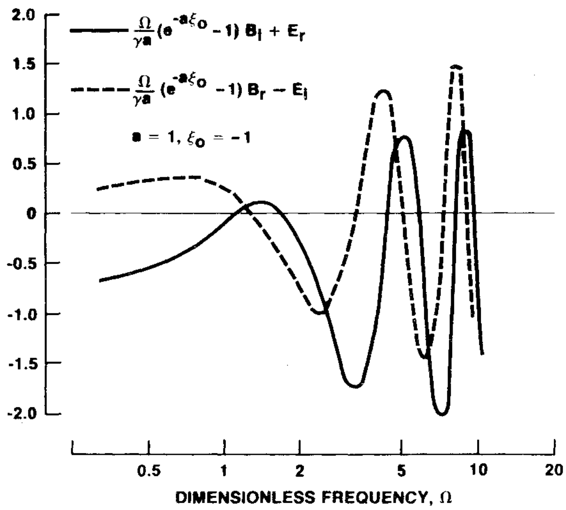

Fig. 2 Coefficient of $(\bar{p}_I^*\bar{T}_0)$ correlations.

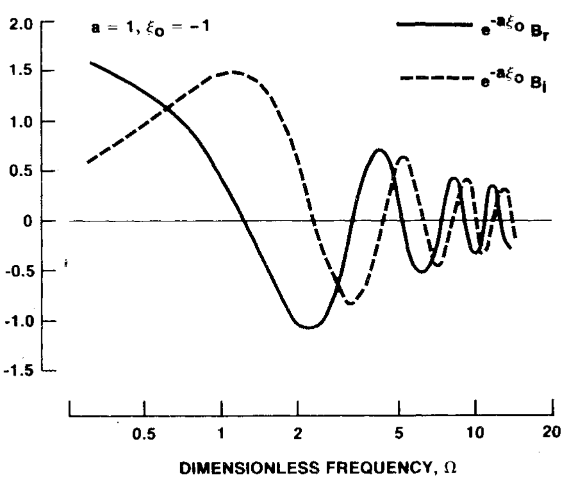

Fig. 3 Coefficient of $(\bar{T}_0^*\bar{u}_I)$ correlations.

$a=1$, $\xi_0=-1$), the coefficient of correlation of temperature fluctuations has a peak value of 0.26 at $\Omega=2.2$, which gives the peak frequency $f=\Omega U_j/(2\pi\ell)=350$ Hz. If the upstream temperature flucuation is 0.5% in one-third octave band at this frequency, then $(\bar{T}_0^*\bar{T}_0)\approx 2.5\times 10^{-5}$. The entropy noise intensity due to this correlation, from Eq. (12) with $P_j=10^5$ N/m^2 is found to be $W=\epsilon^2 U_j P_j (B_r E_r + B_i E_i)\times(\bar{T}_0^*\bar{T}_0)=$ 82.3 W/m^2 or 139.1 dB at the nozzle exit. This is the radiating part of the acoustic energy which should be distinguished from the near-field SPL at the nozzle exit.

The spectrum of the sound pressure square $\bar{p}^2(f)$ at the nozzle exit is obtained by the autocorrelation of $\bar{p}_I(0;\Omega)$ from Eq. (10) in terms of upstream correlations, i.e.,

$$\bar{p}^2(f)=\epsilon^2 P_j^2[(B_r^2+B_i^2)(\bar{T}_0^*\bar{T}_0)+2B_r(\bar{T}_0^*\bar{p}_I)_r$$
$$+2B_i(\bar{T}_0^*\bar{p}_I)_i+(\bar{p}_I^*\bar{p}_I)] \qquad (14)$$

Notice that

$$(\bar{T}_0^*\bar{p}_I)_r=(\bar{p}_I^*\bar{T}_0)_r$$
$$(\bar{T}_0^*\bar{p}_I)_i=-(\bar{p}_I^*\bar{T}_0)_i$$

$(\bar{p}_I^*\bar{p}_I)$ is the spectrum of the upstream sound pressure correlation given by

$$(\bar{p}_I^*\bar{p}_I)=\frac{\ell}{U_j}\frac{2\pi}{\tau_{\text{int}}}\bar{p}_I^*(\xi_0;\Omega)\bar{p}_I(\xi_0;\Omega)$$

from an upstream microphone.

Example 2

For the conditions given in Example 1, the near-field SPL at the nozzle exit due to upstream temperature fluctuation can be calculated from Eq. (14).

$$\bar{p}^2(f)=\epsilon^2 P_j^2(B_r^2+B_i^2)(\bar{T}_0^*\bar{T}_0)$$

If $\Omega=2.2$ or $f=350$ Hz, Fig. 3 gives $e^{-a\xi_0}B_r=-1.1$ and $e^{-a\xi_0}B_i=0.2$, i.e., $B_r^2+B_i^2=0.17$. Therefore, $[\bar{p}^2(f)]^{1/2}=116$ N/m^2. The near-field SPL at the nozzle exit, due to upstream $(\bar{T}_0^*\bar{T}_0)$, is SPL $=20\log_{10}\{[\bar{p}^2(f)]^{1/2}/(2\times10^{-5}$ N/m$^2)\}=135.3$ dB which does not include the upstream noise. In this case, the upstream near-field SPL has to be added to the SPL calculated from $(\bar{T}_0^*\bar{T}_0)$ alone.

The ratio of nozzle entrance area to exit area is e^a. Figures 1-3 show the coefficients for $a=1$. The effect of this area ratio to the nozzle entropy noise can be shown by comparing the coefficients with different values of a. It is found that, for smaller area ratio, the coefficient oscillates at smaller amplitude and the first peak occurs at higher frequency. As a limiting case, a straight duct ($a=0$) generates no entropy noise.

Conclusion

The present analysis of a quasi-one-dimensional flow model illustrates the nozzle entropy noise generation mechanisms. The entropy noise is generated from the upstream inhomogeneities in the accelerated nozzle flow. The quasi-one-dimensional flow model requires the transverse size of the inhomogeneities to be larger than the nozzle diameter. In practical cases, small size inhomogeneities have to be measured across the nozzle entrance and averaged values have to be used.

Upstream temperature fluctuations are considered to be the main source of entropy noise. Results of this analysis show that the nozzle entropy noise is important for the middle-range frequencies, $\Omega\sim O(1)$.

Because the nozzle exit is open to ambient air, the acoustic intensity at the nozzle exit should be used to predict the far-field entropy noise. The calculation of the radiating part of the entropy noise is the main effort of the present analysis.

The upstream correlations of temperature, pressure, and velocity should be measured for a number of subsonic engine and model tests. These measurements would be very important in the development of entropy noise prediction technology in evaluating the entropy noise for engines at low power settings and subsonic hot jet tests.

Appendix A: Zeroth- and First-Order Solutions

The zeroth-order equations are:

$$\frac{\partial\rho_0}{\partial\tau}+\frac{\partial u_0}{\partial\xi}+e^{a\xi}\frac{\partial\rho_0}{\partial\xi}-au_0=0$$

$$\frac{\partial p_0}{\partial\xi}=0$$

$$\frac{\partial T_0}{\partial\tau}+e^{a\xi}\frac{\partial T_0}{\partial\xi}-\frac{\gamma-1}{\gamma}\left(\frac{\partial p_0}{\partial\tau}+e^{a\xi}\frac{\partial p_0}{\partial\xi}\right)=0$$

$$p_0-\rho_0-T_0=0$$

with boundary conditions $p_0(\xi_0,\tau)$, $T_0(\xi_0,\tau)$, and $u_0(\xi_0,\tau)$ as known functions of τ. It is convenient and useful to solve the problem for each frequency. The Fourier transform

$$\bar{P}(\xi;\Omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}P(\xi;\tau)e^{i\Omega\tau}\mathrm{d}\tau$$

is used. The solution to the Fourier-transformed zeroth-order problem (for each dimensionless frequency Ω) is:

$$\bar{p}_0=\bar{p}_0(\xi_0)$$

$$\bar{u}_0=\frac{i\Omega}{\gamma a}\bar{p}_0[e^{a(\xi-\xi_0)}-1]+\bar{u}_0(\xi_0)e^{a(\xi-\xi_0)}$$

$$\bar{T}_0=\frac{\gamma-1}{\gamma}\bar{p}_0+\left\{\bar{T}_0(\xi_0)-\frac{\gamma-1}{\gamma}\bar{p}_0\right\}\exp\left[-\frac{i\Omega}{a}(e^{-a\xi}-e^{-a\xi_0})\right]$$

$$\bar{\rho}_0=\frac{1}{\gamma}\bar{p}_0-\left\{\bar{T}_0(\xi_0)-\frac{\gamma-1}{\gamma}\bar{p}_0\right\}\exp\left[-\frac{i\Omega}{a}(e^{-a\xi}-e^{-a\xi_0})\right]$$

The first-order equations are:

$$\frac{\partial\rho_1}{\partial\tau}+e^{a\xi}\frac{\partial\rho_1}{\partial\xi}+\frac{\partial u_1}{\partial\xi}-au_1=\frac{1}{2}(e^{a\xi}-e^{3a\xi})\frac{\partial\rho_0}{\partial\xi}-ae^{3a\xi}\rho_0$$
$$-\frac{1}{2}(1-e^{2a\xi})\left(\frac{\partial u_0}{\partial\xi}-au_0\right)+ae^{2a\xi}u_0$$

$$\frac{\partial p_1}{\partial\xi}=-\gamma\left[\frac{\partial u_0}{\partial\tau}+e^{a\xi}\frac{\partial u_0}{\partial\xi}+ae^{a\xi}u_0+ae^{2a\xi}\rho_0\right]$$

$$\frac{\partial T_1}{\partial\tau}+e^{a\xi}\frac{\partial T_1}{\partial\xi}-\frac{\gamma-1}{\gamma}\left(\frac{\partial p_1}{\partial\tau}+e^{a\xi}\frac{\partial p_1}{\partial\xi}\right)$$
$$=-\frac{1}{2}(1-e^{2a\xi})\frac{\partial T_0}{\partial\tau}-(\gamma-1)ae^{3a\xi}(T_0-p_0)$$

$$p_1-\rho_1-T_1=\frac{1}{2}(1-e^{2a\xi})[T_0+(\gamma-1)\rho_0]$$

with boundary conditions $p_1(\xi_0,\tau)$, $T_1(\xi_0,\tau)$ and $u_1(\xi_0,\tau)$ as known functions of τ. The solution to the Fourier-transformed first-order problem is:

$$\bar{p}_1 = -\gamma\{ e^{a\xi}\bar{u}_0 + I_2 - [e^{a\xi_0}\bar{u}_0(\xi_0) + I_2(\xi_0)]\} + \bar{p}_1(\xi_0)$$

$$\bar{T}_1 = \frac{\gamma-1}{\gamma}\bar{p}_1 + \exp\left[-\frac{i\Omega}{a}e^{-a\xi}\right][I_{4b}(\xi) - I_{4b}(\xi_0)] + \left\{\bar{T}_1(\xi_0) - \frac{\gamma-1}{\gamma}p_1(\xi_0)\right\}\exp\left[-\frac{i\Omega}{a}(e^{-a\xi} - e^{-a\xi_0})\right]$$

$$\bar{\rho}_1 = \frac{1}{\gamma}\bar{p}_1 - \exp\left[-\frac{i\Omega}{a}e^{-a\xi}\right][I_{4b}(\xi) - I_{4b}(\xi_0)] - \left\{\bar{T}_1(\xi_0) - \frac{\gamma-1}{\gamma}\bar{p}_1(\xi_0)\right\}\exp\left[-\frac{i\Omega}{a}(e^{-a\xi} - e^{-a\xi_0})\right]$$

$$\bar{u}_1 = e^{a\xi}[I_{u_1} - I_{u_1}(\xi_0) + I_{u_2} - I_{u_2}(\xi_0)] - e^{a\xi}[\bar{p}_1 - \bar{p}_1(\xi_0)] + e^{a(\xi-\xi_0)}\bar{u}_1(\xi_0)$$

where I_2, I_{4b}, I_{u1}, and I_{u2} are

$$I_2 = \frac{1}{\gamma}\left\{\frac{1}{2}e^{2a\xi} + \frac{\Omega^2}{a^2}e^{a(\xi-\xi_0)} - \frac{\Omega^2}{a}\xi + (\gamma-1)a \times\exp\left[\frac{i\Omega}{a}e^{-a\xi_0}\right]I_1\right\}\bar{p}_0 - a\exp\left[-\frac{i\Omega}{a}e^{-a\xi_0}\right]\bar{T}_0(\xi_0)I_1 - \frac{i\Omega}{a}u_0(\xi_0)e^{a(\xi-\xi_0)}$$

$$I_1 = \frac{\Omega^2}{2a^2}\left\{\frac{\cos y}{y^2} - \frac{\sin y}{y} + C_i(y) - i\left[\frac{\cos y}{y} + \frac{\sin y}{y^2} + S_i(y)\right]\right\},\quad y = \frac{\Omega}{a}e^{-a\xi}$$

$$C_i(y) = \int_\infty^y \frac{\cos t}{t}\,dt$$

$$S_i(y) = \int_0^y \frac{\sin t}{t}\,dt$$

$$I_{4b} = \frac{\gamma-1}{2\gamma}\bar{p}_0(e^{2a\xi} - 1)\exp\left[\frac{i\Omega}{a}e^{-a\xi}\right] - \frac{1}{2}\exp\left[\frac{i\Omega}{a}e^{-a\xi_0}\right] \times\left[\bar{T}_0(\xi_0) - \frac{\gamma-1}{\gamma}\bar{p}_0\right]\left[\frac{i\Omega}{a}(e^{-a\xi} + e^{a\xi}) + (\gamma-1)e^{2a\xi}\right]$$

$$I_{u1} = \frac{1}{2}(1 - e^{2a\xi})(\bar{p}_0 - \bar{T}_0) + \frac{1}{2}(e^{a\xi} - e^{-a\xi})\bar{u}_0$$

$$I_{u2} = I_3 + I_4 + I_5$$

$$I_3 = -\frac{i\Omega}{\gamma a}e^{-a\xi}\bar{p}_1 - \frac{i\Omega}{a}[\bar{u}_0 + I_{3a} + I_{3b} + I_{3c}]$$

$$I_{3a} = \frac{\Omega^2}{\gamma a}\bar{p}_0\left[e^{-a\xi_0}\xi + \frac{1}{a}e^{-a\xi}\right] - i\Omega e^{-a\xi_0}\bar{u}_0(\xi_0)\xi$$

$$I_{3b} = \frac{i\Omega}{\gamma}\bar{p}_0\left[\frac{1}{a}e^{a(\xi-\xi_0)} - \xi\right] + \bar{u}_0(\xi_0)e^{a(\xi-\xi_0)}$$

$$I_{3c} = \frac{1}{\gamma}e^{a\xi}\bar{p}_0 + \exp\left[\frac{i\Omega}{a}e^{-a\xi_0}\right]\left[\frac{\gamma-1}{\gamma}\bar{p}_0 - \bar{T}_0(\xi_0)\right] \times\left\{\exp\left[a\xi - \frac{i\Omega}{a}e^{-a\xi}\right] - i\Omega I_0\right\}$$

$$I_0 = -\frac{1}{a}[C_i(y) - iS_i(y)],\quad y = \frac{\Omega}{a}e^{-a\xi}$$

$$I_4 = I_{4a} - \exp\left[-\frac{i\Omega}{a}e^{-a\xi}\right][I_{4b}(\xi) - I_{4b}(\xi_0)]$$

$$I_{4a} = \frac{\gamma-1}{2\gamma}\bar{p}_0\left[e^{2a\xi} - \frac{i\Omega}{a}(e^{-a\xi} + e^{a\xi})\right] + \exp\left[\frac{i\Omega}{a}e^{a\xi_0}\right]\left[\bar{T}_0(\xi_0) - \frac{\gamma-1}{\gamma}\bar{p}_0\right] \times\left\{\frac{1}{2}\exp\left[-\frac{i\Omega}{a}e^{-a\xi}\right](1 - e^{2a\xi}) + (2-\gamma)aI_1\right\}$$

$$I_5 = -\left\{\bar{T}_1(\xi_0) - \frac{\gamma-1}{\gamma}\bar{p}_1(\xi_0)\right\}\exp\left[\frac{i\Omega}{a}(e^{-a\xi_0} - e^{-a\xi})\right]$$

Appendix B: Definitions

$$B = -\gamma a\exp\left[\frac{i\Omega}{a}e^{-a\xi_0}\right][I_1(\xi_0) - I_1(0)]$$

$$E = -\frac{\gamma-1}{2}(1 - e^{2a\xi_0}) - \frac{i\Omega}{a}B_3 + \frac{1}{2}B_4 - \frac{1}{\gamma}B$$

$$B_3 = \frac{B}{\gamma} - \exp\left[\frac{i\Omega}{a}e^{-a\xi_0}\right]\left\{\exp\left[-\frac{i\Omega}{a}\right] - \exp\left[a\xi_0 - \frac{i\Omega}{a}e^{-a\xi_0}\right] + i\Omega[I_0(\xi_0) - I_0(0)]\right\}$$

$$B_4 = \exp\left[\frac{i\Omega}{a}(e^{-a\xi_0} - 1)\right] - 1 + \frac{i\Omega}{a}\left\{e^{a\xi_0} - \exp\left[\frac{i\Omega}{a}(e^{-a\xi_0} - 1)\right] - i\Omega\exp\left[\frac{i\Omega}{a}e^{-a\xi_0}\right][I_0(\xi_0) - I_0(0)]\right\} + 2(\gamma-1)a\exp\left[\frac{i\Omega}{a}e^{-a\xi_0}\right][I_1(\xi_0) - I_1(0)]$$

$$I_0(\xi) = -\frac{1}{a}[C_i(y) - iS_i(y)],\quad y = \frac{\Omega}{a}e^{-a\xi}$$

$$C_i(y) = \int_\infty^y \frac{\cos t}{t}\,dt,\quad S_i(y) = \int_0^y \frac{\sin t}{t}\,dt$$

$$I_1(\xi) = \frac{\Omega^2}{2a^3}\left\{\frac{\cos y}{y^2} - \frac{\sin y}{y} + C_i(y) - i\left[\frac{\cos y}{y} + \frac{\sin y}{y^2} + S_i(y)\right]\right\},\quad y = \frac{\Omega}{a}e^{-a\xi}$$

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